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LETTER TO THE EDITOR

Coherent states for parabosons

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Abstract. Taking into consideration that the Fock space of a parabose oscillator may be described by bilinear commutation relations involving creation and annihilation operators, we construct the coherent states for odd and even order parabosons.

Historically the coherent state was first constructed [1] by Schrödinger for a simple harmonic oscillator having an equispaced eigenspectrum. These states became popular and found extensive use in different areas of physics namely, nonlinear optics, laser physics, superfluidity etc [2]. Similar states were later developed by Nieto and Simmons [3] for general potentials having unequal spacing for the eigenspectrum. Efforts have also been made to build a coherently superposed number state for parabose oscillators for which the corresponding creation and annihilation operators satisfy trilinear commutation relations [4]. Recently use has been made of such coherently superposed states to study the nature of the classical motion [5] of the Calogero–Sutherland–Vasiliev oscillator [6] which is presently the object of considerable attention [7]. The common feature of both the Calogero–Sutherland–Vasiliev oscillator and the parabose oscillator is the equispaced eigenspectrum although the number states spanning the Fock space corresponding to them are quite different.

Recently in the context of Green's ansatz Macfarlane [8] has attempted to construct Fock spaces of a parabose oscillator that brings out the significance of bilinear commutation relations rather than the conventional approach based on the trilinear ones. This has made the bosonic structure of the parabose Fock space very transparent. In this article we present a way of constructing the coherent superposition of parabose states by exploiting this new algebraic structure of parabose Fock space.

The order of parabosons can be either odd or even. For the simplest non-trivial case of odd order $p = 3$, the annihilation operator of parabosons can be transformed to [8]

$$\hat{a} = \sqrt{2}(\hat{\alpha}\hat{c} + \hat{\beta}\hat{c}^\dagger) + \hat{\gamma}\hat{c}_3 \tag{1}$$

where $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\gamma}$ are a set of bosonic operators obeying

$$[\hat{\alpha}, \hat{\alpha}^\dagger] = 1 \qquad [\hat{\alpha}, \hat{\alpha}] = 0 \qquad [\hat{\alpha}^\dagger, \hat{\alpha}^\dagger] = 0$$

with similar relations for $\hat{\beta}$, $\hat{\beta}^\dagger$, $\hat{\gamma}$ and $\hat{\gamma}^\dagger$, while the fermionic operators $\hat{c} = \frac{1}{2}\left(\hat{c}_1 - i\hat{c}_2\right)$, $\hat{c}^\dagger = \frac{1}{2}\left(\hat{c}_1 + i\hat{c}_2\right)$ and $\hat{c}_3$ are constrained by

$$[\hat{c}_i, \hat{c}_j] = 2i\epsilon_{ijk}\hat{c}_k \qquad i, j, k = 1, 2, 3.$$

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If the presence of $\hat{\gamma}$ in (1) is ignored, we get Green's ansatz for parabosons of order two. It is easily checked that the operators $\hat{a}$ and $\hat{a}^\dagger$ obey the usual trilinear commutation relation conventionally satisfied by parabosons [9].

A simple realization of $\hat{a}$ is possible by choosing $\hat{c}$, $\hat{c}^\dagger$ and $\hat{c}_3$ as

$$\hat{c} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad \hat{c}^\dagger = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \hat{c}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2}$$

This yields a 2×2 matrix for the annihilation operator of the parabosons

$$\hat{a} = \begin{pmatrix} \hat{\gamma} & \sqrt{2}\hat{\beta} \\ \sqrt{2}\hat{\alpha} & -\hat{\gamma} \end{pmatrix}. \tag{3}$$

From (1) it is clear that the number states of the parabose oscillator can be constructed as the product state of the three bosons and of the fermion and so may be labelled by the number of alpha, beta and gamma bosons and fermions written in that order that is, by $|n_\alpha, n_\beta, n_\gamma; n_c = (0 \text{ or } 1)\rangle$, $n_\alpha, n_\beta, n_\gamma$ and n_c representing the eigenvalues of the corresponding number operators $\hat{N}_\alpha$, $\hat{N}_\beta$, $\hat{N}_\gamma$ and $\hat{N}_c$.

The Hamiltonian for the parabose oscillator, namely

$$H = \tfrac{1}{2}\{\hat{a}, \hat{a}^\dagger\} \tag{4}$$

then becomes

$$H = \tfrac{1}{2}\{\hat{a}, \hat{a}^\dagger\} = \hat{N}_\alpha + \hat{N}_\beta + \hat{N}_\gamma + \tfrac{3}{2} \tag{5}$$

which is identical to that of a three-dimensional oscillator. The conspicuous absence of the fermionic number operator $\hat{N}_c = \hat{c}^\dagger\hat{c}$ in the Hamiltonian may be noted. Further, the parabose states being the product states of more than one bosonic state and a fermionic state there exists scope for greater reducibility. Indeed all the states constructed in this way have degeneracy. As written down explicitly in (5) the degeneracies in the case of odd order parabosons are the same as those for a three-dimensional oscillator while for the even order parabosons they coincide with those of a two-dimensional oscillator.

The subsidiary condition $\hat{a}|0\rangle = 0$ that is to be satisfied by the vacuum state $|0\rangle$ of the parabosons for the complete specification of the underlying Fock space can then be easily accomplished by the identification of the product state $|s, 0, 0; 0\rangle$, $s = 0, 1, 2, \ldots,$ as the vacuum state where the α-oscillator has been displaced to a state of s-quanta. That this would be sufficient for setting up the Fock space of parabosons of odd order $p = (2s+3)$, $s = 0, 1, 2, \ldots,$ is transparent from the observation that the number operator $\hat{N}$ corresponding to the states built up from $|s, 0, 0; 0\rangle$ is given by

$$\hat{N} = H - \tfrac{1}{2}(2s+3) \qquad s = 0, 1, 2, 3, \ldots. \tag{6}$$

It should be remarked that the minimum odd order parabose Fock space describable in this way is three which is not a restriction at all since $p = 1$ parabose corresponds simply to the case of the standard harmonic oscillator.

We now turn to constructing the coherent state for parabosons. Since it is clear that the even order case can be obtained essentially by putting in $\hat{\gamma} = 0$ everywhere in the odd order case it will suffice to carry out the construction for the odd order only. There are essentially two equivalent ways to construct the coherent state. We describe both the methods below. In the first method the displacement operator $e^{Z\hat{a}^\dagger}$ is applied on the vacuum $|0\rangle$. Noting that the explicit expressions for the even and odd states for the parabosons are

$$\begin{aligned} |2n\rangle &\propto \hat{a}^{\dagger 2n}|0\rangle = (2\hat{\alpha}^\dagger\hat{\beta}^\dagger + \hat{\gamma}^{\dagger 2})^n|s, 0, 0; 0\rangle \\ |2n+1\rangle &\propto \hat{a}^{\dagger 2n+1}|0\rangle = (2\hat{\alpha}^\dagger\hat{\beta}^\dagger + \hat{\gamma}^{\dagger 2})^n(\hat{\alpha}^\dagger\hat{c}^\dagger - \hat{\gamma}^\dagger)|s, 0, 0; 0\rangle \end{aligned} \tag{7}$$

it is straightforward to obtain the coherent state $|Z\rangle$ as

$$
\begin{aligned}
|Z\rangle = e^{Z\hat{a}^\dagger}|s,0,0;0\rangle &= \sum_{n=0}^{\infty}\frac{Z^n}{n!}(\hat{a}^\dagger)^n|s,0,0;0\rangle \\
&= \sum_{r=0}^{\infty}\frac{Z^{2r}}{(2r)!}\sum_{m_\beta=0}^{r}\frac{r!2^{m_\beta}}{m_\beta!(r-m_\beta)!}R^s_{r,m_\beta}|s+m_\beta,m_\beta,2r-2m_\beta;0\rangle \\
&\quad+\sum_{q=0}^{\infty}\frac{Z^{2q+1}}{(2q+1)!}\sum_{m'_\beta=0}^{q}\frac{q!2^{m'_\beta}}{m'_\beta!(q-m'_\beta)!} \\
&\quad\times\Big[\sqrt{2(m'_\beta+s+1)}R^s_{q,m'_\beta}|s+m'_\beta+1,m'_\beta,2q-2m'_\beta;1\rangle \\
&\quad-\sqrt{(2q-2m'_\beta+1)}R^s_{q,m'_\beta}|s+m'_\beta,m'_\beta,2q-2m'_\beta+1;0\rangle\Big]
\end{aligned}
\tag{8}
$$

where

$$
R^s_{q,m_\beta} = \sqrt{\frac{(m_\beta+s)!m_\beta!(2q-2m_\beta)!}{s!}}.
$$

It should be remarked that $|Z\rangle$ contains both zero and one fermionic states though the one fermionic state is only associated with the odd para states.

On the other hand, the second construction asks for the coherent state $|Z\rangle$ to be an eigenstate of $\hat{a}$ with eigenvalue Z:

$$
\hat{a}|Z\rangle = Z|Z\rangle. \tag{9}
$$

Defining the coherent state $|Z\rangle$ as the product of the simple bosonic coherent states $|z_\alpha\rangle$, $|z_\beta\rangle$, $|z_\gamma\rangle$ corresponding to the respective bosonic operators and realizing that it would be a column matrix in the fermionic representation we have

$$
\hat{a}|Z\rangle = \begin{pmatrix} \hat{\gamma} & \sqrt{2}\hat{\beta} \\ \sqrt{2}\hat{\alpha} & -\hat{\gamma} \end{pmatrix}\begin{pmatrix} |z_\alpha,z_\beta,z_\gamma\rangle \\ |z'_\alpha,z'_\beta,z'_\gamma\rangle \end{pmatrix} = Z\begin{pmatrix} |z_\alpha,z_\beta,z_\gamma\rangle \\ |z'_\alpha,z'_\beta,z'_\gamma\rangle \end{pmatrix}. \tag{10}
$$

This at once gives rise to the condition

$$
(\hat{\gamma}^2+2\hat{\alpha}\hat{\beta})|z_\alpha,z_\beta,z_\gamma\rangle = Z^2|z_\alpha,z_\beta,z_\gamma\rangle \tag{11}
$$

and a similar one for $|z'_\alpha,z'_\beta,z'_\gamma\rangle$. Thus it is seen that the bosonic part of $|Z\rangle$ is an eigenstate of $(\hat{\gamma}^2+2\hat{\alpha}\hat{\beta})$ with eigenvalue Z^2. For the individual bosonic states $|z_i\rangle$, $i=\alpha,\beta,\gamma$ we take the standard coherent states built up from their respective groundstates $|n_i=0\rangle$, $i=\alpha,\beta,\gamma$ as an eigenstate of the corresponding annihilation operators:

$$
|z_i\rangle = (\exp(-|z_i|)^2/2)\sum_{n_i=0}^{\infty}\frac{{z_i}^{n_i}}{\sqrt{n_i!}}|n_i\rangle \qquad i=\alpha,\beta,\gamma. \tag{12}
$$

so that

$$
|Z\rangle = \begin{pmatrix} \sum_{n_\alpha,n_\beta,n_\gamma}\left(\prod_{i=\alpha,\beta,\gamma}\frac{z_i^{n_i}}{\sqrt{n_i!}}\right)|n_\alpha,n_\beta,n_\gamma;0\rangle \\ \sum_{n'_\alpha,n'_\beta,n'_\gamma}\left(\prod_{i=\alpha,\beta,\gamma}\frac{z_i^{n'_i}}{\sqrt{n'_i!}}\right)|n'_\alpha,n'_\beta,n'_\gamma;1\rangle \end{pmatrix}. \tag{13}
$$

By construction therefore $|Z\rangle$ is an eigenstate of $(\hat{\gamma}^2+2\hat{\alpha}\hat{\beta})$ and we have by virtue of (11)

$$
Z^2 = 2z_\alpha z_\beta + z_\gamma^2. \tag{14}
$$

We now conveniently parametrize $z_\alpha, z_\beta, z_\gamma$ as

$$
z_\alpha = \lambda_\alpha\,\mathrm{e}^{\mathrm{i}(\theta+\phi)} \qquad z_\beta = \lambda_\beta\,\mathrm{e}^{\mathrm{i}(\theta-\phi)} \qquad z_\gamma = \lambda_\gamma\,\mathrm{e}^{\mathrm{i}\theta} \tag{15a}
$$

so that

$$Z^2 = (2\lambda_\alpha\lambda_\beta + \lambda_\gamma^2)\,e^{2i\theta}. \tag{15b}$$

Incorporating these we can write the expression for the coherent state in terms of λs, θ and ϕ. Note that the parameters of the primed ket $|z'_\alpha, z'_\beta, z'_\gamma\rangle$ also satisfy similar relations such as (14) and (15).

The coherent state for the parabosons should be built from the vacuum state which we have seen to correspond to a product bosonic state with the α-bosonic state being displaced to a state of s-quanta. This can be conveniently incorporated in our above mentioned state (13) if we observe that the operator $\hat{\Lambda}$ defined as

$$\hat{\Lambda} = \hat{\alpha}^\dagger\hat{\alpha} - \hat{\beta}^\dagger\hat{\beta} - \hat{c}^\dagger\hat{c} = \begin{pmatrix} \hat{\alpha}^\dagger\hat{\alpha} - \hat{\beta}^\dagger\hat{\beta} & 0 \\ 0 & \hat{\alpha}^\dagger\hat{\alpha} - \hat{\beta}^\dagger\hat{\beta} - 1 \end{pmatrix} \tag{16}$$

commutes with $\hat{a}$ as well as with $(\hat{\gamma}^2 + 2\hat{\alpha}\hat{\beta})$. Hence we can require the coherent state to be a simultaneous eigenstate of these operators. Thus the states which satisfy

$$\hat{\Lambda}|n_\alpha, n_\beta, n_\gamma; 0\rangle = s|n_\alpha, n_\beta, n_\gamma; 0\rangle \qquad \hat{\Lambda}|n'_\alpha, n'_\beta, n'_\gamma; 1\rangle = (s+1)|n'_\alpha, n'_\beta, n'_\gamma, 1\rangle \tag{17}$$

can only be superposed to have the desired coherent state for the parabosons. To carry out this excercise, we project out the states of definite value s of $\hat{\Lambda}$ from $|Z\rangle$ in the following way [10]

$$|Z; s\rangle = \frac{1}{2\pi}\int_0^{2\pi} d\phi\, e^{-i(s+\hat{N}_c)\phi}|Z\rangle. \tag{18}$$

We then obtain

$$|Z; s\rangle \equiv |\lambda_\alpha, \lambda_\beta, \lambda_\gamma, \theta; s\rangle$$
$$= \begin{pmatrix} \sum_{n_\beta, n_\gamma} \frac{\lambda_\alpha{}^{n_\beta+s}\lambda_\beta{}^{n_\beta}\lambda_\gamma{}^{n_\gamma}}{\sqrt{(n_\beta+s)!(n_\beta)!(n_\gamma)!}} e^{i\theta(2n_\beta+n_\gamma+s)}|n_\beta+s, n_\beta, n_\gamma; 0\rangle \\ \sum_{n'_\beta, n'_\gamma} \frac{\lambda_\alpha{}^{n'_\beta+s+1}\lambda_\beta{}^{n'_\beta}\lambda_\gamma{}^{n'_\gamma}}{\sqrt{(n'_\beta+s+1)!(n'_\beta)!(n'_\gamma)!}} e^{i\theta(2n'_\beta+n'_\gamma+s+1)}|n'_\beta+s+1, n'_\beta, n'_\gamma; 1\rangle \end{pmatrix}. \tag{19}$$

Substituting now the condition that the paraboson number states $|n\rangle$ are odd or even according to whether $n_\alpha + n_\beta + n_\gamma = n$ are odd or even, we finally obtain the paraboson coherent state as

$$|Z; s\rangle = \begin{pmatrix} \sum_{m=\beta}^{\infty}\sum_{n_\beta=0}^{m} \frac{(\lambda_\alpha\lambda_\beta e^{2i\theta})^{n_\beta}(\lambda_\gamma^2 e^{2i\theta})^{m-n_\beta}}{\sqrt{(n_\beta+s)!(2m-2n_\beta)!(n_\beta)!}}|n_\beta+s, n_\beta, 2m-2n_\beta; 0\rangle \\ \sum_{m=0}^{\infty}\sum_{n_\beta=0}^{m} \frac{(\lambda_\alpha e^{i\theta})(\lambda_\alpha\lambda_\beta e^{2i\theta})^{n_\beta}(\lambda_\gamma^2 e^{2i\theta})^{m-n_\beta}}{\sqrt{(n_\beta+s+1)!(2m-2n_\beta)!(n_\beta)!}}|n_\beta+s+1, n_\beta, 2m-2n_\beta; 1\rangle \end{pmatrix} \tag{20}$$

with the parametrization $Z^2 = (2\lambda_\alpha\lambda_\beta + \lambda_\gamma^2)\,e^{2i\theta}$. It may be noted that the first element in the expression corresponds to the superposition of even states $|2n\rangle \propto |r+s, r, 2n-2r; 0\rangle$ while the second element to that of odd states $|2n+1\rangle \propto |n_\beta+s+1, n_\beta, 2n-2n_\beta; 1\rangle$. The comparison with the states obtained by Sharma *et al* [4] cannot be made directly as we have remarked earlier that this formalism has greater scope for reducibility because of the introduction of more than one boson in the problem. It may be remarked that the coherent states for even order parabosons can be readily obtained by putting $z_\gamma = 0$ and working with $Z^2 = 2z_\alpha z_\beta$.

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